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A weakly mixing tiling dynamical system with a smooth model☆

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Abstract

We describe a weakly mixing one-dimensional tiling dynamical system in which the tiling space is modeled by a surface M of genus 2. The tiling system satisfies an inflation, and the inflation map is modeled by a pseudo-Anosov diffeomorphism D on M. The expansion coefficient θ for D is a non-Pisot number. In particular, the leaves of the expanding foliation for D are tiled by their visits to the elements of a Markov partition for D. The tiling dynamical system is an almost 1:1 extension of the unit speed flow along these leaves.

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1. Introduction

In this paper, we construct an example of a weakly mixing one-dimensional tiling dynamical system $(\tilde{\mathscr{V}}, \sigma^t)$ (cf. [3,10,11]) that satisfies an inflation map E with an expansion coefficient θ that is not a Pisot number. By a result of Solomyak [11] it follows that σ^t is a weakly mixing flow. The tiling space $\tilde{\mathscr{V}}$ is modeled by a surface M of genus 2, and the inflation E is an almost 1:1 extension of a pseudo-Anosov

☆ This work was carried out in conjunction with the dissertation [5] of the first author.

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diffeomorphism D on M with expansion θ. The tiling flow σ^t is an almost 1:1 extension of the unit speed flow along the nonsingular leaves of the expanding foliation for D. The tiling flow σ^t is obtained as a suspension of a weakly mixing interval exchange transformation.

The example in this paper should be compared to the two-dimensional Penrose tiling dynamical system (cf. [10]), which is modeled by a group of rotations on the 4-torus $\mathbf{T}^4$. The Penrose inflation is an almost 1:1 extension of an Anosov diffeomorphism on $\mathbf{T}^4$. In the Penrose case, the expansion is a Pisot number, and the tiling dynamical system has pure point spectrum. Other weakly mixing tiling dynamical systems with non-Pisot expansions have been studied (cf. [3,11]), but so far, no concrete model for any of them has been found.

The tiling dynamical system constructed here is one-dimensional, but a two-dimensional weakly mixing example can be obtained by talking the Cartesian square (cf. [3]) of the given one-dimensional example. In that case, the smooth model is a four manifold: the Cartesian square of the surface of genus 2. Unfortunately, this two-dimensional example has nonergodic directions.

When we say the tiling space $\tilde{\mathscr{V}}$ is modeled by the genus 2 surface M, we mean that there exists a continuous surjection $\psi : \tilde{\mathscr{V}} \to M$ that is 1:1 on the preimages of nonsingular leaves of the expanding foliation for D. This expanding foliation has one 6-pronged singularity with total angle 6π. In the tiling space $\tilde{\mathscr{V}}$, this singular leaf "splits" into six different tilings, which pair off according to their common past or common future. Thus, typical points on the singular leaf have 2 preimages, but the singular point itself has 6 preimages.

The explicit construction of a pseudo-Anosov diffeomorphism with a non-Pisot expansion is of some independent interest, Arnoux [2] constructed a pseudo-Anosov diffeomorphism on a surface of genus 3 with a Pisot expansion coefficient θ (satisfying $\theta^3 - \theta^2 - \theta - 1 = 0$). He showed that this example is an almost 1:1 extension of an Anosov diffeomorphism on the 3-torus. In earlier work [1] he constructed, for each $g \geqslant 2$, a pseudo-Anosov diffeomorphism on a surface of genus g with an expansion that is Pisot of degree g.

The example in this paper was obtained by using the general recipe of Veech [14] for constructing pseudo-Anosov diffeomorphisms. In this method, based on the theory of Rauzy induction [9], the surface M is the phase space of a suspension of a self-inducing interval exchange transformation. The construction described here requires for its initiation, an interval exchange transformation T that induces itself on a subinterval whose length is a non-Pisot fraction θ of the length of the original interval. The use of Rauzy Induction to find such an example T is described in [5].

2. The interval exchange transformation *T*

We begin by defining an interval exchange transformation T on an interval J. Recall that a *Pisot number* is a real algebraic integer $\theta > 1$, all of whose conjugates have modulus less than 1 [12]. We show that this transformation T "induces" itself on a subinterval J' of J, where the ratio $|J|/|J'|$ is not a Pisot number.

Let

$$A=\begin{pmatrix}1&1&1&1\\1&2&0&0\\0&0&2&1\\2&3&2&2\end{pmatrix}. \tag{1}$$

Since $A^2>0$, there exists a unique Perron–Frobenius eigenvalue $\theta>0$, larger than the modulus of its conjugates, and corresponding eigenvector $\vec{\lambda}>0$. In this example, $\theta=\frac{1}{4}(7+\sqrt{5}+\sqrt{2}\sqrt{19+7\sqrt{5}})\approx 4.39026$ and $\vec{\lambda}\approx(0.47726, 0.19967, 0.41836, 1)$. It can be shown that θ is a non-Pisot number.

Using $\vec{\lambda}$ and the permutation $\pi=(14)(23)$, we define the interval exchange transformation $T:J\to J$, where $J=[0,\sum_{i=1}^{4}\lambda_i)$. In particular, the interval J is divided into four subintervals J_1, J_2, J_3, J_4 of widths λ_1, λ_2, λ_3, λ_4, respectively, which are left closed and right open. The transformation T maps J onto itself by cutting J into these four subintervals and rearranging their order according to the permutation π.

Now we define the induced transformation T' on the subinterval $J'=[0,(1/\theta)\sum_{i=1}^{4}\lambda_i)$ of J. Note that $\sum_{i=1}^{4}\lambda_i=\theta\lambda_1$, so $J'=[0,\lambda_1)=J_1$. For each $s\in J'$, let $n(s)>0$ be the smallest positive integer such that $T^{n(s)}(s)\in J'$. Then, $T':J'\to J'$ is defined by $T's=T^{n(s)}s$.

Lemma 2.1. *(J',T') is conjugate to (J,T).*

Proof. Let us partition J' into four subintervals:

$$J_1'=[0,4\lambda_1+\lambda_2-2\lambda_4),$$

$$J_2'=[4\lambda_1+\lambda_2-2\lambda_4, 2\lambda_1+\lambda_2-\lambda_4),$$

$$J_3'=[2\lambda_1+\lambda_2-\lambda_4, 3\lambda_1+2\lambda_2+\lambda_3-2\lambda_4),$$

$$J_4'=[3\lambda_1+2\lambda_2+\lambda_3-2\lambda_4, \lambda_1).$$

We will show that $n(s)$ is constant on these intervals. Note that the subintervals J_i' have lengths λ_i' where $\vec{\lambda}'=A^{-1}\vec{\lambda}=(1/\theta)\vec{\lambda}$. We build a tower of intervals (Fig. 1) by stacking the images of a subinterval above the subinterval itself, until the iterates return to J'. For example,

$$T(J_1')=T([0,4\lambda_1+\lambda_2-2\lambda_4))=(\lambda_2+\lambda_3+\lambda_4, 4\lambda_1+2\lambda_2+\lambda_3-\lambda_4)\subset J_4,$$

$$T^2(J_1')=T([\lambda_2+\lambda_3+\lambda_4, 4\lambda_1+2\lambda_2+\lambda_3-\lambda_4))$$

$$=[-\lambda_1+\lambda_4, 3\lambda_1+\lambda_2-\lambda_4)\subset J_2,$$

$$T^3(J_1')=[-2\lambda_1+\lambda_3+2\lambda_4, 2\lambda_1+\lambda_2+\lambda_3)\subset J_4,$$

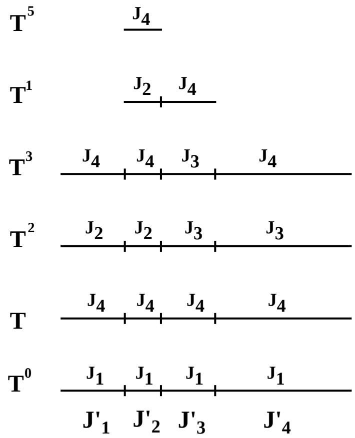

Fig. 1. Return tower of J'.

and

$$T^4(J_1') = [-3\lambda_1 - \lambda_2 + 2\lambda_4, \lambda_1) \subset J'.$$

It follows that $n(s) = 4$ for all $s \in J_1'$. So, the tower above J_1' has four levels, which we label J_1, J_4, J_2, J_4. By a similar calculation, one can show $n(s) = 6$ for all $s \in J_2'$, $n(s) = 5$ for all $s \in J_s'$, and $n(s) = 4$ for all $s \in J_4'$.

To show that (J', T') is conjugate to (J, T), we define $g : J \to J'$ by $g(s) = (1/\theta)s$. Obviously, g is one-to-one, onto, and continuous. One can verify that g satisfies the commutation relation $g \circ T = T' \circ g$. We show this for $s \in J_1$. We have

$$g(Ts) = g(s + \lambda_2 + \lambda_3 + \lambda_4) = \frac{1}{\theta}(s + \lambda_2 + \lambda_3 + \lambda_4) = \frac{1}{\theta} s + \frac{1}{\theta}(\lambda_2 + \lambda_3 + \lambda_4)$$

and

$$T'(g(s)) = T'\left(\frac{1}{\theta} s\right) = T^4\left(\frac{1}{\theta} s\right) = \frac{1}{\theta} s - 3\lambda_1 - \lambda_2 + 2\lambda_4.$$

But $(1/\theta)(\lambda_2 + \lambda_3 + \lambda_4) = -3\lambda_1 - \lambda_2 + 2\lambda_4$ by summing the last three coordinates of $A^{-1}\vec{\lambda} = (1/\theta)\vec{\lambda}$. A similar argument is used for $s \in J_2, J_3, J_4$.

The conjugacy $g \circ T = T' \circ g$ shows that T' is also an interval exchange defined by the permutation π. $\square$

An interval exchange transformation T on J is said to be *minimal*, if the orbit $\mathcal{O}(s) = \{T^i s \colon i \in \mathbf{N}\}$ of each point $s \in J$ is dense in J [14].

Lemma 2.2. *The coordinates of $\vec{\lambda}$ are rationally independent.*

Proof. Let $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \{1, \sqrt{5}, \sqrt{2}\sqrt{19 + 7\sqrt{5}}, \sqrt{10}\sqrt{19 + 7\sqrt{5}}\}$. By an explicit calculation, we can write $\lambda_i = c_{i1}\omega_1 + c_{i2}\omega_2 + c_{i3}\omega_3 + c_{i4}\omega_4$ for each $i = 1, 2, 3, 4$

where $c_{ij} \in \mathbf{Q}$. The numbers $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ are easily seen to be rationally independent. Thus, $\vec{\lambda} = \vec{\omega}C$, where $\vec{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ and $C = (c_{ij})$. If $\vec{\lambda} \cdot \vec{n} = 0$, then $(c_{i1}, c_{i2}, c_{i3}, c_{i4}) \cdot \vec{n} = 0$ for each i, since the coordinates of $\vec{\omega}$ are rationally independent. Written in matrix form $C\vec{n}^{\mathrm{t}} = 0$, which implies $\vec{n} = 0$ since C is invertible. □

Corollary 2.1. *T is minimal on J.*

Proof. By Keane [7], this follows from the fact that π is irreducible (i.e., $\pi(\{1, 2, \ldots, j\}) \neq \{1, 2, \ldots, j\}$ for any $j = 1, 2, 3$), and from Lemma 2.2. □

3. The substitution system X

In this section, we construct a substitution system on an alphabet $\mathscr{A} = \{1, 2, 3, 4\}$. Let $\mathscr{A}^*$ be the set of finite words in the alphabet $\mathscr{A}$, and let $\mathscr{A}^{\mathbf{N}}$ be the set of (one-sided) sequences with values in $\mathscr{A}$, endowed with the product topology. We define a substitution map $S : \mathscr{A} \to \mathscr{A}^*$ by

$$1 \to 1424, \quad 2 \to 142424, \quad 3 \to 14334, \quad 4 \to 1434.$$

This substitution depends on the interval exchange T. It records the subintervals of J which lie over each J_i' as shown in Fig. 1. In particular, if $i \to i_1 i_2, \ldots, i_k$, then the levels over J_i are labeled $J_{i_1}, J_{i_2}, \ldots, J_{i_k}$ in the tower.

By a standard construction (cf. [8]), we can extend S to a map $S : \mathscr{A}^{\mathbf{N}} \to \mathscr{A}^{\mathbf{N}}$. The map S has a unique fixed point $u \in \mathscr{A}^{\mathbf{N}}$, and we define X to be the orbit closure of u in $\mathscr{A}^{\mathbf{N}}$ under the shift map $\sigma : \mathscr{A}^{\mathbf{N}} \to \mathscr{A}^{\mathbf{N}}$.

Lemma 3.1. *(X, σ) is uniquely ergodic.*

This follows from [8] since the substitution matrix A, defined in (1), is primitive.

Lemma 3.2. *There exists an injective, right-continuous mapping $\phi : J \to \mathscr{A}^{\mathbf{N}}$ such that $\phi \circ T = \sigma \circ \phi$ (so that $\phi(J)$ is σ-invariant), and $\phi(J)$ is dense in X.*

Proof. We define a mapping $\phi : J \to \mathscr{A}^{\mathbf{N}}$ by

$$(\phi(s))_i = k \quad \text{where } T^i s \in J_k \text{ for each } i \in \mathbf{N}. \tag{2}$$

Since each subinterval J_k is open on the right, ϕ is right continuous.

Now, suppose there exist two points $s_1, s_2 \in J$ with $s_1 < s_2$ and $\phi(s_1) = \phi(s_2)$. Then, $s_1, s_2 \in J_n$ for some $n = 1, 2, 3, 4$ since $(\phi s_1)_0 = (\phi s_2)_0$. Let c_1, c_2, c_3, c_4 be the left endpoints of the subintervals of J. By Corollary 2.1, we know T^{-1} is also minimal. Thus, there exists $n_i \geqslant 0$ with $T^{-n_i} c_i$ in the open interval (s_1, s_2) for each $i = 1, 2, 3, 4$. Let c_k be the first of these endpoints $T^{-n_k} c_k \in (s_1, s_2)$. Since T is 1:1, $T^{n_k} s_1 \neq T^{n_k} s_2$. Suppose $T^{n_k} s_1 < T^{n_k} s_2$, which can occur when $k = 2, 3$, or 4. Then, $(\phi s_1)_{n_k} = 1, 2, \ldots$, or $k - 1$ and $(\phi s_2)_{n_k} = k, k + 1, \ldots$, or 4. Suppose $T^{n_k} s_1 > T^{n_k} s_2$, which can occur when $k = 1, 2$, or 3. Then, $(\phi s_1)_{n_k} = 4$ and $(\phi s_2)_{n_k} = 1, 2$, or 3. This shows that $(\phi s_1)_{n_k} \neq (\phi s_2)_{n_k}$, which is a contradiction.

To show $\phi \circ T = \sigma \circ \phi$, we show $(\phi(Ts))_i = (\sigma(\phi s))_i$ for all $s \in J$ and $i \in \mathbf{N}$. Suppose $(\phi(Ts))_i = k$. Then, $T^i(Ts) = T^{i+1}s \in J_k$, so $k = (\phi s)_{i+1} = (\sigma(\phi s))_i$, and $\phi(J)$ is σ-invariant.

Finally, we need only show that $\phi(J) \subset X$. First we show $\phi(0) = u$, the unique fixed point of the substitution S. Define $\phi' : J' \to \mathscr{A}^{\mathbf{N}}$ by

$$(\phi'(s))_i = k \quad \text{where } (T')^i s \in J'_k \text{ for each } i \in \mathbf{N}.$$

Since (J, T) is conjugate to (J', T') by $g : J \to J'$, we have $\phi(s) = \phi'(g(s))$ for all $s \in J$.

We claim $S(\phi'(s)) = \phi(s)$ for all $s \in J'$, which we show by induction on the nth digit of $\phi'(s)$. The substitution S records the subintervals of J the orbits of $s \in J'$ visit, until it returns to J'. We separate $s \in J'$ into four cases:

$$s \in J'_1 \;\Rightarrow\; S((\phi' s)_0) = S(1) = 1424 = (\phi s)_{[0,n(s)-1]},$$

$$s \in J'_2 \;\Rightarrow\; S((\phi' s)_0) = S(2) = 142\,424 = (\phi s)_{[0,n(s)-1]},$$

$$s \in J'_3 \;\Rightarrow\; S((\phi' s)_0) = S(3) = 14\,334 = (\phi s)_{[0,n(s)-1]},$$

$$s \in J'_4 \;\Rightarrow\; S((\phi' s)_0) = S(4) = 1434 = (\phi s)_{[0,n(s)-1]}.$$

Assuming $S((\phi' s)_{[0,m-1]}) = (\phi s)_{[0,n^m(s)-1]}$, we show

$$S((\phi' s)_{[0,m]}) = (\phi s)_{[0,n^{m+1}(s)-1]}.$$

Note that $T's = T^{n(s)}s \in J'$ for all $s \in J'$, so we can write $(T')^m s = T^{n^m(s)}s$ for all $s \in J'$ and $m \in \mathbf{N}$. We apply $(T')^m s$ in each case above to obtain $S((\phi'(T')^m s)_0) = (\phi T^{n^m(s)} s)_{[0,n(s)-1]}$. But $S((\phi'(T')^m s)_0) = S((\phi' s)_m)$ and $(\phi T^{n^m(s)} s)_{[0,n(s)-1]} = (\phi s)_{[n^m(s), n^{m+1}(s)-1]}$. Using this fact and the assumption, we have $S((\phi' s)_{[0,m]}) = (\phi s)_{[0,n^{m+1}(s)-1]}$ which proves the claim.

Finally, we note that $S(\phi(0)) = S(\phi'(g(0))) = S(\phi'(0)) = \phi(0)$. Consequently, $\phi(0)$ is the fixed point of S, so $\phi(0) = u$. Thus, 0 and all of its iterates $\{T^i 0\}$ have their ϕ-images in X. By minimality of T, there exists sequence $\{n_j\} > 0$ with $\{T^{n_j} 0\} \searrow s$. Thus, by right continuity of ϕ, we have $\{\phi(T^{n_j} 0)\} = \{\sigma^{n_j}(\phi(0)\} = \{\sigma^{n_k} u\} \to \phi(s)$, so $\phi(s) \in cl(\mathcal{O}(u)) = X$. $\square$

Note that ϕ is also left continuous on J except at the pre-images of endpoints of the subintervals of J. Let $J^o = J \setminus \bigcup_{i=0}^{\infty} T^{-i}\{0, \lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3\}$ be the subset of J which excludes all preimages of endpoints of the subintervals.

Theorem 3.1. *There exists a continuous mapping $\rho : X \to J$ which is onto, almost* 1:1, *and at most* 2:1. *Also, $\rho^{-1} = \phi$ on J, and ρ satisfies the commutation relation $\rho \circ \sigma = T \circ \rho$ on $X \backslash \rho^{-1}\{\lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3\}$. In particular, $card\{\rho^{-1}(s)\} = 2$, for $s \in J \backslash J^o$.*

Proof. Define the map $\rho : X \to J$ by $\rho(\sigma^k u) = T^k 0$ for each $k \in \mathbf{N}$. We claim ρ is uniformly continuous on $\mathcal{O}(u)$. Consider $x = \sigma^k u$ for some $k \in \mathbf{N}$. Then, for each $l \in \mathbf{N}$ we have $x = \cdot u_k u_{k+1} \dots u_{k+l+1} \dots$, which means $\rho(\sigma^k u) = T^k(0) \in J_{u_k}$. So, $T \circ \rho(\sigma^k u) = T^{k+1}(0) \in J_{u_{k+1}}$, which gives $\rho(\sigma^k u) \in T^{-1}(J_{u_{k+1}})$. It follows that $\rho(\sigma^k u) \in T^{-l}(J_{u_{k+l}})$. Thus, we can write

$$\rho(\sigma^k u) \in \bigcap_{j=0}^{l} T^{-j}(J_{u_{k+j}}).$$

For each $l \in \mathbf{N}$, define

$$\varepsilon_l = \max_{[u_k \dots u_{k+l}]} \left\{ \operatorname{diam} \left[\bigcap_{j=0}^{l} T^{-j}(J_{u_{k+j}}) \right] \right\},$$

where the max is taken over all words $[u_k \dots u_{k+l}]$ of length $l+1$. By minimality of T, $\varepsilon_l \to 0$ as $l \to \infty$.

Let $\varepsilon > 0$. Choose $L > 0$ such that $\varepsilon_l < \varepsilon$ for $l \geqslant L$. Then, for any $\sigma^{k_1} u, \sigma^{k_2} u \in \mathcal{O}(u)$ with $u_{k_1} \dots u_{k_1+l} = u_{k_2} \dots u_{k_2+l}$ we have $\rho(\sigma^{k_1} u), \rho(\sigma^{k_2} u) \in \bigcap_{j=0}^{l} T^{-j}(J_{u_{k_1+j}})$. But this set has diameter $\varepsilon_l < \varepsilon$, so $|\rho(\sigma^{k_1} u) - \rho(\sigma^{k_2} u)| < \varepsilon$. Therefore, we can extend ρ to a continuous map on X since $\mathcal{O}(u)$ is dense in X.

To prove $\rho = \phi^{-1}$, it suffices to show $\rho(\phi(s)) = s$ on J. Let $s \in J$. Since T is minimal, there exists an increasing sequence $\{n_i\}$ such that $\{T^{n_i}(0)\} \searrow s$. Since ϕ is right continuous, $\{\phi(T^{n_i}(0))\} \to \phi(s)$. But $\phi(T^{n_i}(0)) = \sigma^{n_i} u$ for each n_i, so $\{\sigma^{n_i} u\} \to \phi(s)$. Therefore, by continuity of ρ,

$$\rho(\phi(s)) = \rho\left(\lim_{n_i \to \infty} \sigma^{n_i} u \right) = \lim_{n_i \to \infty} \rho(\sigma^{n_i} u) = \lim_{n_i \to \infty} T^{n_i}(0) = s$$

by definition of s. $\square$

Theorem 3.2. *The transformation (J, T) is uniquely ergodic.*

This follows from Lemmas 3.1, 3.2, and Theorem 3.1.

In Section 6, we are going to consider the future and past images of $s \in J$. For this reason it will be important to extend 1-sided substitution space $X \subset \mathscr{A}^{\mathbf{N}}$ to a 2-sided substitution space $\bar{X} \subset \mathscr{A}^{\mathbf{Z}}$ (cf. [8]). In particular, if we extend ϕ to a map $\bar{\phi} : J \to \mathscr{A}^{\mathbf{Z}}$, then $\bar{X}$ is the closure of $\bar{\phi}(J)$. This extension is unique for ϕ-images of points in J that are not forward images under T of any discontinuities of T.

For points with nonunique left extensions there exist exactly two left extensions in $\bar{X}$. Let $\bar{\rho} : \bar{X} \to J$ be the extension of ρ to $\bar{X}$. Then Theorem 3.1 holds with X and ρ replaced by $\bar{X}$ and $\bar{\rho}$, and J^o replaced by the set of all forward and backward iterates of the discontinuities of T on J.

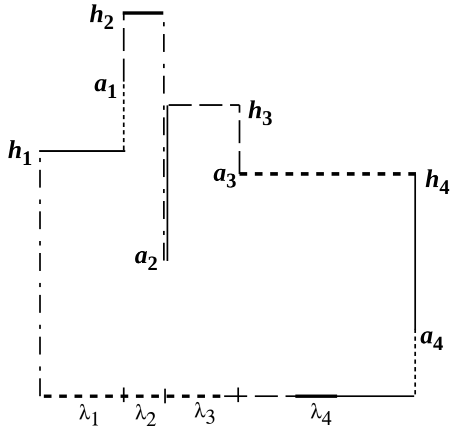

Fig. 2. Surface M.

4. The surface M

In this section we construct a diffeomorphism D on a surface M, defined as the phase space of the suspension of the interval exchange transformation T on J, under a height function $\hat{h}: J \to \mathbf{R}$. Note that the Perron–Frobenius eigenvalues of A and A^t are the same, namely θ. We define the function $\hat{h}$ in terms of the Perron–Frobenius eigenvector $\vec{h}$ of A^t by $\hat{h}(s) = h_i$ for $s \in J_i$, $i = 1, 2, 3, 4$. Here $\vec{h} \approx (1.09529, 1.71333, 1.29496, 1)$.

We define the surface M by

$$M = \{(s, y) \in \mathbf{R}^2 \colon \ s \in J, \ \ y \in [0, \hat{h}(s))\}$$

with the following identifications. We eliminate the horizontal edges by identifying the points $(s, \hat{h}(s))$ and $(Ts, 0)$. This is illustrated by the different horizontal lines in Fig. 2. We introduce "zippers" to define the identifications of the vertical edges of the polygon to obtain a closed surface. The zippers begin at each of the corners along the top of the polygon and unzip the vertical edges down to a specified height given by $\vec{a} \in \mathbf{R}^4$. This is illustrated by different types of vertical lines in Fig. 2. In particular, for the vertical identifications to be invariant with respect to the map D, to be defined below, it will suffice for the vector $\vec{a}$ to be the solution to the matrix equation $L\vec{h} + \vec{a} = \theta\vec{a}$, where the matrix L is given by

$$L = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This follows from Veech [14], who introduced the idea of zippers and described the relation between the matrices L and A. In our example, $\vec{a} \approx (1.41836, 0.61803, 1, 0.32307)$.

We are going to define a hyperbolic map D on M by a construction reminiscent of the baker's transformation. First, we construct a partition $\mathcal{Q}$ of M by dividing M into 11 rectangles as shown on the left side of Fig. 3. The partition $\mathcal{Q}=\{Q_1,Q_2,\dots,Q_{11}\}$ is defined by $Q_i=M\cap([\beta_{i-1},\beta_i)\times\mathbf{R})$, for $i=1,2,\dots,11$, where

$$\begin{aligned}
\beta_0 &= 0, & \beta_6 &= \lambda_1+\lambda_2+\lambda_3,\\
\beta_1 &= \lambda_1, & \beta_7 &= -\lambda_1-\lambda_2+2\lambda_4,\\
\beta_2 &= -\lambda_1+\lambda_4, & \beta_8 &= \lambda_3+\lambda_4,\\
\beta_3 &= \lambda_1+\lambda_2, & \beta_9 &= -2\lambda_1+\lambda_3+2\lambda_4,\\
\beta_4 &= 2\lambda_1+2\lambda_2+\lambda_3-\lambda_4, & \beta_{10} &= \lambda_2+\lambda_3+\lambda_4,\\
\beta_5 &= \lambda_4, & \beta_{11} &= \lambda_1+\lambda_2+\lambda_3+\lambda_4.
\end{aligned}$$

Using this partition, we restack the 11 rectangles using $J'=J_1$ as the new base interval. This new surface, denoted M', is shown in the center of Fig. 3. We denote this operation by $D_1: M\to M'$.

Lemma 4.1. *The identifications of M are preserved in M'.*

Proof. That the horizontal identifications are preserved follows from the fact that T induces itself on J_1. Thus, we can stack the rectangles of the partition $\mathcal{Q}$ above Q_1 to construct M' so that the tops of lower rectangles agree with the bases of stacked rectangles as shown.

To see that the vertical identifications are preserved, we label the vertical sides of each member of $\mathcal{Q}$ by the letters $\{\mathrm{a},\mathrm{b},\dots,\mathrm{m}\}$ as shown on the first construction of M. It suffices to show that each lettered side adjoins the same rectangles of $\mathcal{Q}$ on the two polygons.

Consider the vertical edge labeled a in the first polygon, namely M. Edge a is the right edge of Q_3 above edge f and the left edge of Q_1. This is also true for the middle polygon, representing M'. Considering edge b in both diagrams, it adjoins Q_1 on its left to Q_2 on its right. This argument can be repeated for each vertical edge. Therefore, each rectangle in $\mathcal{Q}$ is identified in the same way. □

Now we define $D_2: M'\to M$ by $D_2(s,y)=(\theta s,(1/\theta)y)$. Finally, let $D: M\to M$ by $D=(D_2\circ D_1)^{-1}$.

Lemma 4.2. *The map D is a homeomorphism on the surface M.*

This follows from Lemma 4.1.

5. The pseudo-Anosov diffeomorphism D

A *measured foliation with singularities* $\mathcal{F}$ on a surface M is a set of leaves such that for every regular (nonsingular) point of M there exists a neighborhood and a

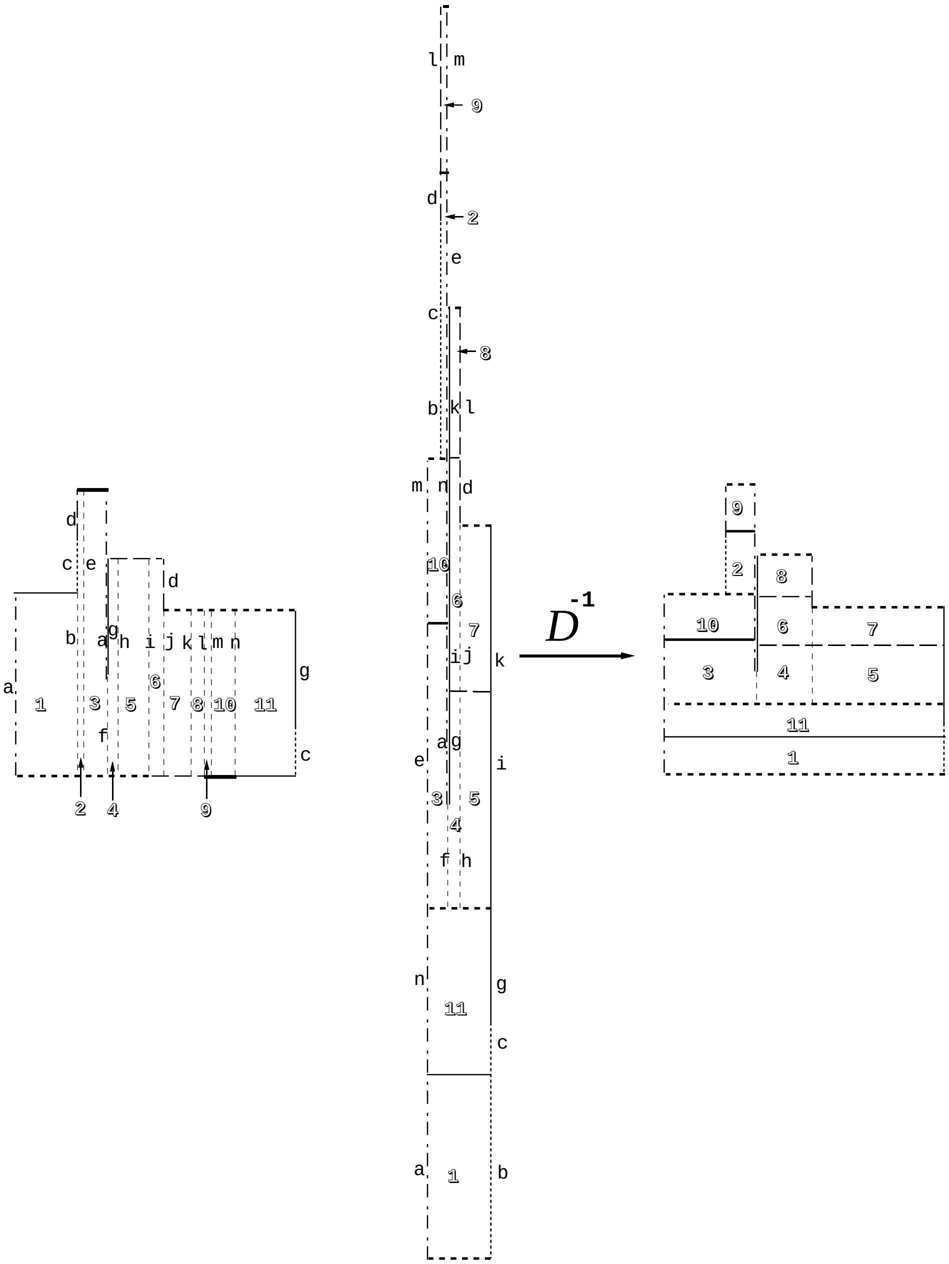

Fig. 3. The map $D^{-1}: M \rightarrow M$.

mapping that sends the leaves in the neighborhood to horizontal lines in $\mathbf{R}^2$ [13]. These mappings must be isometries on the leaves. A finite number of "singularities" are permitted, which may be p-pronged saddles with $p \geqslant 3$.

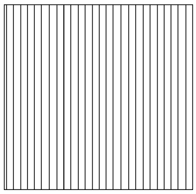

Fig. 4. Neighborhood of regular points of foliation $\mathscr{F}$.

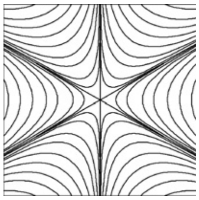

Fig. 5. Neighborhood of the singularity of foliation $\mathscr{F}$.

Consider the foliation $\mathscr{F}$ of M consisting of vertical lines along M, where M is again viewed as a polygon, as in Fig. 2. All regular points of M have neighborhoods which look like Fig. 4. It is easy to see that the singularity, located at the bottom of the longest zipper, is 6-pronged, or equivalently a *branch point* of the surface M of order 3. This is shown in Fig. 5. All other points on M have a total angle of 2π, so there is only one branch point of M. We can compute the genus of M by using

$$2g(M) - 2 = \sum_{\text{branch points}} (\text{order of branch point} - 1),$$

(cf. [14]). Thus, we have $g(M) = 2$.

A *pseudo-Anosov diffeomorphism* D is a homeomorphism of a surface such that there exists a number $\theta > 1$ and a pair of transverse measurable foliations $\mathscr{F}^s$ and $\mathscr{F}^u$ with $D(\mathscr{F}^s) = (1/\theta)\mathscr{F}^s$ and $D(\mathscr{F}^u) = \theta\mathscr{F}^u$ [13]. This means, in particular, that D preserves the two foliations $\mathscr{F}^s$ and $\mathscr{F}^u$, and contracts the leaves of $\mathscr{F}^s$ by $1/\theta$ and expands the leaves of $\mathscr{F}^u$ by θ.

Theorem 5.1. *The homeomorphism D is a pseudo-Anosov diffeomorphism of M.*

Proof. The partition $\mathscr{F}$ into vertical lines is $\mathscr{F}^u$. The analogous partition into horizontal lines is $\mathscr{F}^s$. □

A *Markov partition* [6] for D on M is a collection $\mathscr{P} = \{P_1, \dots, P_k\}$ of rectangles (i.e., closed subsets whose four sides are segments of the stable and unstable foliations) such that:

1. $\bigcup_{i=1}^k P_i = M$;
2. $\text{int}(P_i) \cap \text{int}(P_j) = \emptyset$ for all $i \neq j$;

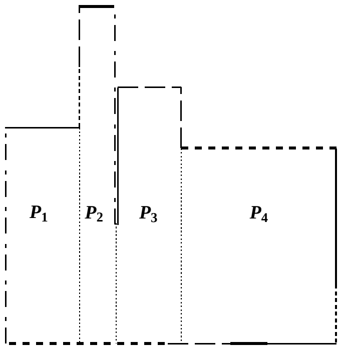

Fig. 6. Markov Partition $\mathscr{P}$.

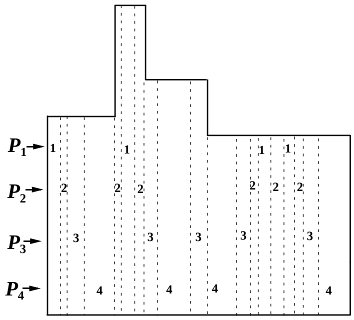

Fig. 7. $D(\mathscr{P})$.

3. If $z \in \mathrm{int}(P_i)$ and $D(z) \in \mathrm{int}(P_j)$, then $D(\mathscr{F}^s(z, P_i)) \subset \mathscr{F}^s(D(z), P_j)$ and $D(\mathscr{F}^u(z, P_i)) \supset \mathscr{F}^u(D(z), P_j)$.

We note that Markov partitions always exist for pseudo-Anosov diffeomorphisms [4].

Let $(h_i) = [0, h_i]$ where h_i the ith entry of the Perron-Frobenius eigenvector $\vec{h}$ of A^t. We define partition $\mathscr{P} = \{P_1, P_2, P_3, P_4\}$ of M where P_i is the closure of $J_i \times (h_i)$. This partition of M is shown in Fig. 6.

Lemma 5.1. *$\mathscr{P}$ is a Markov partition of M for D.*

This follows from Figs. 7 and 8.

Given a pseudo-Anosov diffeomorphism D on an orientable surface M, we define a flow F^t on the surface M as follows. First we fix an orientation for the expanding foliation $\mathscr{F}^u$ for D on M. Then, we define F^t to be the flow that moves with unit speed in a positive direction along the leaves of $\mathscr{F}^u$. Note that F^t in not well defined on the singular leaf of $\mathscr{F}^u$, but it is defined Lebesgue almost everywhere.

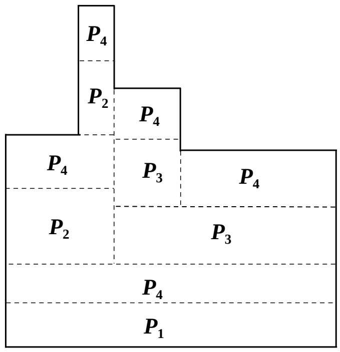

Fig. 8. $D^{-1}(\mathscr{P})$.

In our example, F^t is the suspension of T with the return time function $\hat{h}$. The trajectory of a point in the flow F^t will be used, in the next section to define a tiling.

6. The tiling dynamical system $\tilde{\mathscr{V}}$

A tiling of $\mathbf{R}$ $\mathscr{U}=\{(h_i)_k)\}=\{(h_i)+s_{k,i}\}$, is a collection of intervals such that $s_{k,i}\in\mathbf{R}$ and $\mathbf{R}=\bigcup_k (h_i)_k$, where the union is essentially disjoint (overlaps only at the endpoints).

Let $\tilde{\mathscr{U}}$ be a translation invariant set of tilings. We provide $\tilde{\mathscr{U}}$ with the *tiling topology* (cf. [10,11]). In particular, two tilings are close if, up to a small translation, they agree on a large interval around the origin. One can show that the *tiling space* $\tilde{\mathscr{U}}$ is compact and metrizable (cf. [5]), and the action of $\mathbf{R}$ on $\tilde{\mathscr{U}}$ by translation is continuous. We call a closed, translation invariant subspace $\tilde{\mathscr{V}}\subseteq\tilde{\mathscr{U}}$, together with the action σ^t of $\mathbf{R}$ by translation, a *tiling dynamical system*.

Let $\theta\in\mathbf{R}$. A tiling $\mathscr{U}$ is θ-*subdividing* if for each $U\in\mathscr{U}$, θU is a union of tiles in $\mathscr{U}$, and also $U=t+U'\Leftrightarrow\theta U=\theta t+\theta U'$ for some $t_1,t_2\in\mathbf{R}$. A new tiling $E(\mathscr{U})$, called the *inflation* of $\mathscr{U}$, is defined to be the collection of all the tiles in the union of θU for all $U\in\mathscr{U}$. A tiling of $\mathbf{R}$ is called *self-similar* with expansion θ (cf. [11]), if it is θ-subdividing, has a *finite number of local patterns*, and has the *local isomorphism property*. The finite local patterns condition always holds for one-dimensional tilings with finitely many tile types. The local isomorphism property is equivalent to a tiling being an almost periodic point in a tiling dynamical system (cf. [10]). A one-dimensional tiling space $\tilde{\mathscr{V}}$ is said to *satisfy an inflation* if it is the orbit closure of a self-similar tiling. In this case the inflation E extends to continuous mapping on $\tilde{\mathscr{V}}$.

The tilings of interest here are determined by the flow F^t. Let

$$M^o=M\backslash\{(s,y)\colon\ F^t(s,y)\in\{0,\lambda_1,\lambda_1+\lambda_2,\lambda_1+\lambda_2+\lambda_3\}\ \text{for some}\ t\in\mathbf{R}\}$$

be the surface M with the singular leaf removed. We define the mapping $\tau: M^o\to\tilde{\mathscr{U}}$ by $\tau(s,y)(t)=p(F^t(s,y))$ where the mapping $p: M^o\to\{1,2,3,4\}$ determines the partition

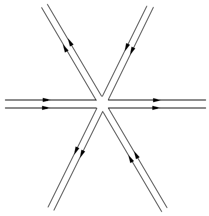

Fig. 9. "Doubling" the singular leaf.

element containing (s,t). Note that τ is 1:1 since the coding map ϕ defined in (2) is 1:1. Thus, we have $\tau(F^t(s,y))=\sigma^t(\tau(s,y))$ for all $t\in\mathbf{R}$ and $(s,y)\in M^o$.

Recall that F^t is not defined on the singular leaf, and thus τ is not defined there. To fix this, we are going to construct a new space by replacing the interval J in the suspension construction with the two-sided substitution space $\bar{X}$. In particular, we define a real-valued function $\bar{h}$ on $\bar{X}$ by $\bar{h}(x)=h_{x_0}=\hat{h}(\bar{\rho}(x))$. We define $\bar{F}^t$ to be the suspension of σ on $\bar{X}$ under the function $\bar{h}$, and we denote the phase space for this suspension by $\bar{M}=\{(x,s)\colon 0\leqslant x<\bar{h}(x)\}$ with appropriate identifications. We think of $\bar{M}$ as a "new surface". The flow $\bar{F}^t$ is defined everywhere on $\bar{M}$ because the shift on $\bar{X}$ is a homeomorphism.

We define $\psi:\bar{M}\to M$ by $\psi(x,s)=(\bar{\rho}(x),s)$. Away from the singular leaf ψ is 1:1 and it and intertwines F^t and $\bar{F}^t$. This is because $\bar{X}$ is an almost 1:1 extension of T on J via the factor map $\bar{\rho}$. Since the singularity of $\mathscr{F}$ is 6-pronged, we have that ψ^{-1} applied to the singular leaf consists of six separate orbits of $\bar{F}^t$ (see Fig. 9). Points on the prongs have two preimages, and the singular point itself has six preimages.

We define the tiling dynamical system $\tilde{\mathscr{V}}$ to be the orbit closure of any tiling $\mathscr{V}=\tau(s,y)=\tau^*(\phi(s),y)$ such that $(s,y)\in M^o$. Here $\tau^*:\bar{M}\to\tilde{\mathscr{U}}$ denotes the extension of τ to $\bar{M}$, defined in terms of the Markov partition induced on $\bar{M}$ by $\mathscr{P}$. Equivalently, $\tilde{\mathscr{V}}$ is the orbit closure of the tiling $\mathscr{V}=\tau^*(u^*,0)$, where $u^*\in\bar{X}$ is the unique two-sided extension of the unique fixed point $u\in X$ of the one-sided substitution S.

By construction τ^* is a homeomorphism of $\bar{M}$ onto $\tilde{\mathscr{V}}$, satisfying $\tau^*\circ\bar{F}^t=\sigma^t\circ\tau^*$. It follows that τ^* is a topological conjugacy, and consequently we identify the flow $(\bar{M},\bar{F}^t)$ to the tiling dynamical system $(\tilde{\mathscr{V}},\sigma^t)$. We have thus proved the following theorem.

Theorem 6.1. *There exists a continuous mapping $\psi:\tilde{\mathscr{V}}\to M$ that is* 1:1 *on the dense G_δ set $\psi^{-1}(M^o)\subset\tilde{\mathscr{V}}$. Elsewhere, the mapping ψ is either* 2:1 *or* 6:1. *Moreover, the restriction of ψ to $\psi^{-1}(M^o)$ conjugates the tiling dynamical system $(\tilde{\mathscr{V}},\sigma^t)$ and the flow (M^o,F^t).*

We say the flow F^t on M is a *smooth model* for the tiling dynamical system.

Now we investigate some of the ergodic properties of the tiling dynamical system $(\tilde{\mathscr{V}},\sigma^t)$.

Lemma 6.1. *The tiling $\mathscr{V}$ is self-similar with expansion θ.*

Proof. The tiling $\mathscr{V}$ has the local isomorphism property since it is a point in the suspension of a minimal one-dimensional substitution. If $\mathscr{V}=\bigcup_k (h_i)_k=\bigcup_k (h_i)+s_{k,i}$, then $\theta\mathscr{V}=\bigcup_k(\theta h_i)+\theta s_{k,i}$. For the substitution S write $S(i)=i_1 i_2,\dots,i_k$, and define

$$E((\theta h_i))=(h_{i_1})\cup(h_{i_2})+h_{i_1}\cup\cdots\cup(h_{i_k})+\sum_{j=1}^{k-1}h_{i_j}.$$

Finally, define the inflation map by

$$E(\mathscr{V})=\bigcup_k E((\theta h_i))+\theta s_{k,i}.$$

Then $E(\mathscr{V})=\mathscr{V}$. □

Theorem 6.2. *The tiling dynamical system* $(\tilde{\mathscr{V}},\sigma^t)$, *defined as the orbit closure of* $\mathscr{V}$, *is minimal and uniquely ergodic. Moreover, it satisfies an inflation with an inflation map E that is an almost* 1:1 *extension of the pseudo-Anosov diffeomorphism D.*

Proof. The first statement follows from Lemma 6.1 and the general theory of tiling dynamical systems in [11]. The second statement follows from the discussion preceding Theorem 6.1. □

Let μ denote the unique invariant Borel measure for $\tilde{\mathscr{V}}$.

Theorem 6.3. *The tiling dynamical system* $\tilde{\mathscr{V}}$ *is weakly mixing, but not strongly mixing.*

Proof. Strongly mixing fails since the tiling space satisfies an inflation (cf. [11]). Weak mixing follows from the fact that the expansion coefficient θ is non-Pisot [11]. □

7. The dynamics of measured foliations

We can use the preceding results to understand the dynamics of the unit speed flow along the expanding leaves for the pseudo-Anosov diffeomorphism D on M. However, this result is actually more general; it applies to an arbitrary pseudo-Anosov diffeomorphism of an oriented surface.

Theorem 7.1. *Let D be a pseudo-Anosov diffeomorphism of an oriented surface M. Let* $\mathscr{F}$ *be the expanding (or contracting) foliation with a given orientation, and let* θ *be the corresponding expansion (or contraction) coefficient. Let* F^t *be the unit speed flow in the positive direction on the nonsingular leaves of* $\mathscr{F}$. *The flow* F^t, *which is defined Lebesgue a.e. on M, is minimal (i.e., every orbit is dense) and Lebesgue uniquely ergodic. It is not strongly mixing, but is weakly mixing if and only if* θ *is not Pisot.*

Proof. We find a Markov partition $\mathscr{P}$ for D and construct the corresponding tiling dynamical system. $\square$

For a pseudo-Anosov diffeomorphism D on an oriented surface M, it follows from Veech [14] that the map induced on a segment $U \subset \mathscr{F}^s$ by the flow F^t along $\mathscr{F}^s$ is always an interval exchange transformation T. With due care in the choice of U, one can insure T is self-inducing. Thus, as noted by Veech [14], any pseudo-Anosov diffeomorphism D on an oriented surface M can be obtained as a suspension of a self inducing interval exchange transformation.

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